$$\int \sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} - \int x \cdot \frac{1}{2}\frac{-2x}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$$

ここで，二項目に関して，$x = a\sin\theta$ と置くと，

$dx = a\cos\theta d\theta$

$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos^2\theta,$

よって

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \int \frac{a^2\sin^2\theta}{a\cos\theta} \cdot a\cos\theta d\theta$$

$$= a^2\int \sin^2\theta d\theta = \frac{a^2}{2}\int (1-\cos 2\theta)d\theta$$

$$= \frac{a^2}{2}\int d\theta - \frac{a^2}{2}\int \cos 2\theta d\theta = \frac{a^2}{2}\theta - \frac{a^2}{2} \cdot \frac{1}{2}\sin 2\theta$$

$$= \frac{a^2}{2}\theta - \frac{a^2}{2}\sin\theta\cos\theta = \frac{a^2}{2}\theta - \frac{a^2}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= \frac{a^2}{2}\theta - \frac{a^2}{2} \cdot \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} = \frac{a^2}{2}\theta - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

以上から，

$$\int \sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$